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L_p Metrics for Compact, Convex Sets

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1. INTRODUCTION

In the active area of approximation of convex sets (see, for instance, the extensive survey of Gruber [6]), a useful device is to identify sets with their support functions and to proceed in function-theoretic terms. The correspondence between the Hausdorff metric and the L_∞ distance between support functions can be exploited in this way (Weil [10], McClure and Vitale [8], Davis, Vitale, and Ben-Sabar [4], Kenderov [7]). Other L_p metrics can be defined, as well, and used for approximation (McClure and Vitale [8]). Related work on the pth means of support functions appears in Firey [5]. The purpose of this note is to establish some results relating these metrics and, in particular, to connect the L_p $(1 \leqslant p < \infty)$ metrics with the more widely studied L_∞ metric. Our motivation comes from some of the work cited above and also the study of *random* sets (e.g., Artstein [1], Baddeley [2], Vitale [9]). The L_2 metric is attractive in this context because of the well-developed spectral theory of random functions. We mention too that certain aspects of Davis [3] are examples of L_2 approximation.

In the next section, we set some notation and preliminaries. Section 3 contains our main quantitative result (Theorem 2) which gives tight bounds between the L_p $(1 \leqslant p < \infty)$ and L_∞ metrics. In the last section, we use this result to show that the derived metric spaces are closely related and that the analogue of the classical Blaschke selection theorem holds for each (Theorem 3).

2. PRELIMINARIES

We shall work in R^d for arbitrary but fixed d, $2 \leqslant d \leqslant \infty$. The Euclidean norm and inner product will be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. On the unit sphere S^{d-1}, we impose unit Lebesgue measure $\mu(\cdot)$.

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$\mathscr{K}^d$ will stand for the space of non-empty compact, convex subsets of R^d. To each $K \in \mathscr{K}^d$, we assign a support function $S_K \in C(S^{d-1})$ via

$$S_K(e) = \max_{x \in K} \langle e, x \rangle, \qquad e \in S^{d-1}.$$

It is Lipschitz continuous and uniquely paired to K. Other properties are

$$K \subseteq L \Leftrightarrow S_K \leqslant S_L$$

$$S_{\mathrm{conv}\{K \cup L\}} = \max\{S_K, S_L\}$$

$$S_{K+\{x\}} = S_K + \langle x, \cdot \rangle.$$

The *Hausdorff metric* between K and L is

$$\delta_\infty(K, L) = \max\{\sup_{x \in K} \inf_{x' \in L} |x - x'|, \sup_{x' \in L} \inf_{x \in K} |x - x'|\}$$

and is equal to $\sup_{e \in S^{d-1}} |S_K(e) - S_L(e)| = \|S_K - S_L\|_\infty$. The other L_p metrics are defined directly on the support functions:

$$\delta_p(K, L) = \left(\int_{S^{d-1}} |S_K(e) - S_L(e)|^p \, \mu(de) \right)^{1/p}, \qquad 1 \leqslant p < \infty.$$

3. An Inequality for Support Functions

In comparing the metrics, there is an immediate bound in one direction.

Theorem 1. *Let $K, L \in \mathscr{K}^d$. Then $\delta_p(K, L) \leqslant \delta_\infty(K, L)$. Equality is attained iff one set is a parallel body of the other.*

Proof. The inequality is direct. For equality, $\delta_p(K, L) = \delta_\infty(K, L) \Leftrightarrow |S_K - S_L| \equiv \max|S_K - S_L| \Leftrightarrow S_K - S_L \equiv \text{const}$. Thus if $S_K = S_L + \rho$, where $\rho > 0$, K is the parallel body to L at radius ρ.

In the other direction, we shall see that there is no bound of the form $C \cdot \delta_\infty \leqslant \delta_p$ for a universal positive constant. While this is familiar from the general theory of L_p spaces, it is not immediate in the restricted class of support functions. Indeed, we shall repeatedly see that the particular properties of support functions lead to quite specialized and often stronger results than are true in general. In large part, it will be convenient to proceed by exploiting geometrical arguments.

Our main quantitative result is a complement to Theorem 1 and establishes a tight bound in the other direction. From it, we will derive our later results.

THEOREM 2. *Let $K, L \in \mathscr{K}^d$ with $D = \operatorname{diam}(K \cup L)$. Then*

$$C_p(K, L) \cdot \delta_\infty(K, L) \leqslant \delta_p(K, L), \qquad 1 \leqslant p < \infty$$

where

$$C_p(K, L) = \left[\frac{1}{B(\frac{1}{2}, (d-1)/2) \sin^p(\theta_0)} \int_0^{\theta_0} \sin^p(\theta_0 - \theta) \sin^{d-2}\theta \, d\theta \right]^{1/p},$$

$B(\cdot, \cdot)$ is the beta integral,

$$\begin{aligned} \sin \theta_0 &= 1/\gamma, & 1 \leqslant \gamma \leqslant \sqrt{4/3} \\ &= 1/\sqrt{1 + \gamma^2/4}, & \sqrt{4/3} \leqslant \gamma, \end{aligned}$$

and

$$\gamma = D/\delta_\infty(K, L).$$

Equality is achieved when the two sets are respectively a disk and its convex hull with a point on its normal axis of symmetry.

Before proceeding to the proof, we note that equality can be achieved for other pairs of sets. Their description is complicated, however, and, in any case, the pair indicated is minimal in a sense suggested by the reduction arguments of the proof.

Proof. An outline of the proof is as follows. Under the conditions

(C1) $\delta_\infty(K, L) = 1$,

(C2) $\operatorname{diam}(K \cup L) \leqslant D$ ($D \geqslant 1$ necessarily),

we derive $\inf \delta_p(K, L)$. This will be done by starting with a pair (K, L) and successively modifying it through five steps $(K_1, L_1), \ldots, (K_5, L_5)$ so that (C1) and (C2) are maintained and $\delta_p(K_i, L_i)$ decreases. At step 5, the geometry is simple enough to work analytically, and we calculate the infimum. Finally, the normalization (C1) is removed to produce the general result.

We begin with $K, L \in \mathscr{K}^d$ satisfying (C1) and (C2). Without loss of generality, assume that $1 = \delta_\infty(K, L) =$ distance from $x_0 \in L$ to K.

Step 1. Let $K_1 = K$ and $L_1 = \operatorname{conv}\{K \cup L\}$ so that $K_1 \subseteq L_1$. Recall that $S_{L_1} = \max\{S_{K_1}, S_L\}$ so that

$$0 \leqslant S_{L_1} - S_{K_1} \leqslant |S_L - S_K|$$

and hence $\delta_p(K_1, L_1) \leqslant \delta_p(K, L)$.

Step 2. Let $K_2 = K_1$ and set $L_2 = \operatorname{conv}\{x_0, K_1\}$. We have $K_1 \subseteq L_2 \subseteq L_1$ or $S_{K_1} \leqslant S_{L_2} \leqslant S_{L_1}$ so that $\delta_p(K_2, L_2) \leqslant \delta_p(K_1, L_1)$.

Step 3. Without loss of generality, assume that $0 \in K_2$ and $\|x_0\| = 1$. We symmetrize about the line through the origin and x_0: let $\{\mathscr{C}_\alpha\}_A$ be the subgroup of orthogonal transformations leaving x_0 invariant and let $m(\cdot)$ be associated normalized Haar measure. Set K_3 and L_3 to be the respective averaged sets

$$K_3 = \int_A \mathscr{C}_\alpha K_2 m(d\alpha)$$

$$L_3 = \int_A \mathscr{C}_\alpha L_2 m(d\alpha).$$

It is clear that $\delta_\infty(K_3, L_3) = 1$ since $\mathscr{C}_\alpha x_0 = x_0$ $\forall \alpha \in A$. For (C2), recall that $\text{diam}(K') = \max_{e \in S^{d-1}} S_{K'}(e) + S_{K'}(-e)$ for any set K. Set $K'_2 = \text{conv}\{K_2 \cup L_2\}$ and $K'_3 = \text{conv}\{K_3 \cup L_3\}$. Then

$$K'_3 = \int_A \mathscr{C}_\alpha K'_2 m(d\alpha)$$

and

$$\begin{aligned}
\text{diam}(K'_3) &= \max_{e \in S^{d-1}} S_{K'_3}(e) + S_{K'_3}(-e) \\
&= \max_{e \in S^{d-1}} \left[\int_A S_{\mathscr{C}_\alpha K'_2}(e)\, m(d\alpha) + \int_A S_{\mathscr{C}_\alpha K'_2}(-e)\, m(d\alpha) \right] \\
&\leqslant \int_A \max_{e \in S^{d-1}} [S_{\mathscr{C}_\alpha K'_2}(e) + S_{\mathscr{C}_\alpha K'_2}(-e)]\, m(d\alpha) \\
&\leqslant \int_A \text{diam}(\mathscr{C}_\alpha K'_2)\, m(d\alpha) = \text{diam}(K'_2).
\end{aligned}$$

Finally, observe that $\delta_p(K, L)$ is a convex functional of $S_K - S_L$ and so decreases upon averaging the sets.

Step 4. Let $L_4 = L_3$ and set $K_4 = \{x \in \mathbb{R}^d \mid x \in L_4 \text{ and } \langle x, x_0 \rangle \leqslant 0\}$. The effect of this is to enlarge K_3 to K_4 $(\subseteq L_4)$ which has a (possibly degenerate) circular face F in the plane $\langle x, x_0 \rangle = 0$. The verifications are similar to those for steps 1 and 2.

Step 5. Set $K_5 = F$ and $L_5 = \text{conv}\{x_0, F\}$. Note that, for $\langle e, x_0 \rangle \geqslant 0$,

$$S_{L_5}(e) - S_{K_5}(e) = S_{L_5}(e) - S_F(e) = S_{L_4}(e) - S_{K_4}(e)$$

and, for $\langle e, x_0 \rangle \leqslant 0$,

$$S_{L_5}(e) - S_{K_5}(e) = 0 = S_{L_4}(e) - S_{K_4}(e).$$

Hence, $S_{L_5} - S_{K_5} \equiv S_{L_4} - S_{K_4}$ and $\delta_p(K_5, L_5) = \delta_p(K_4, L_4)$.

This concludes the reductions. We recapitulate the final picture: K_5 is a disk F in the $\langle x, x_0 \rangle = 0$ plane, and L_5 is the convex hull of F with the point x_0 ($\|x_0\| = 1$).

Next we recast the integral

$$I = \delta_p^p(K_5, L_5) = \int_{S^{d-1}} [S_{L_5}(e) - S_{K_5}(e)]^p \, \mu(de)$$

by integrating around circles, i.e., sets of constant $\langle e, x_0 \rangle$. The integrand vanishes, as mentioned above for $e \ni \langle e, x_0 \rangle \leqslant 0$ and indeed for $e \ni \langle e, x_0 \rangle \leqslant S_F(e)$. Put another way, the integrand vanishes for $\langle e, x_0 \rangle \leqslant \cos \theta_0$ where $\tan \theta_0 = 1/R$, R being the radius of F. For $\langle e, x_0 \rangle = \cos \theta \geqslant \cos \theta_0$, the integrand is $(\cos \theta - \sin \theta / \tan \theta_0)^p$. The (infinitesimal) fraction of points e on S^{d-1} with $\langle e, x_0 \rangle = \cos \theta = \alpha$ is $\varphi(\alpha) \, d\alpha = B_d(1 - \alpha^2)^{(d-3)/2} \, d\alpha$, $B_d = \Gamma(d/2)/\sqrt{\pi} \, \Gamma((d-1)/2)$. I can then be rewritten as

$$\int_{\theta \geqslant \theta_0} \left(\cos \theta - \frac{\sin \theta}{\tan \theta_0} \right)^p \varphi(\alpha) \, d\alpha = \frac{B_d}{\sin^p \theta_0} \int_0^{\theta_0} \sin^p(\theta_0 - \theta) \sin^{d-2}\theta \, d\theta.$$

It is a simple exercise to show that this is an increasing function of θ_0 or, equivalently, a decreasing function of R (recall $\tan \theta_0 = 1/R$). Hence, to minimize, we should take R as large as possible under the constraint $\operatorname{diam}(K_5 \cup L_5) \leqslant D$. Direct trigonometry shows that if $1 \leqslant D \leqslant \sqrt{4/3}$, we should take $R = \sqrt{D^2 - 1}$; in this case the diameter of $K_5 \cup L_5$ is achieved between x_0 and a point on the boundary of F. If $\sqrt{4/3} \leqslant D$, we take $R = D/2$; in this case, the diameter is between opposite points on the boundary of F. In terms of maximizing θ_0, we have

$$\begin{aligned} \sin \theta_0 &= 1/D, & 1 \leqslant D \leqslant \sqrt{4/3} \\ &= 1/\sqrt{1 + D^2/4}, & \sqrt{4/3} \leqslant D. \end{aligned}$$

Inserting those values and taking a pth root yields the result for the special case $\delta_\infty(K, L) = 1$.

It remains to observe that for arbitrary K and L, it is enough to apply the last result to $K' = 1/\delta_\infty(K, L) \cdot K$ and $L' = 1/\delta_\infty(K, L) \cdot L$.

4. Some Consequences

The expression for $C_p(K, L)$ shows that it tends to zero as θ_0 tends to zero. This occurs when the point on the axis of symmetry of the disk tends toward the center of the disk, or, more generally, when $\delta_\infty(K, L)/\text{diam}(K \cup L) \to 0$. Hence, there is no universal positive constant which can be inserted into the inequality. Nevertheless, we shall find that the metrics are very closely related (Theorem 3).

We begin by producing a more transparent (and necessarily looser) form of the inequality.

Corollary 1. *Let $K, L \in \mathscr{K}^d$ with $D = \text{diam}(K \cup L)$. Then*

$$C'_p(K, L) \cdot [\delta_\infty(K, L)]^{(p+d-1)/p} \leqslant \delta_p(K, L)$$

where

$$C'_p(K, L) = \left[\frac{B(p+1, d-1)}{B(\frac{1}{2}, (d-1)/2) \cdot D^{d-1}} \right]^{1/p}.$$

Proof. The result follows from inserting the inequality

$$\begin{aligned} \alpha_{d-2,p}(\theta_0) &= \int_0^{\theta_0} \sin^p \theta \sin^{d-2}(\theta_0 - \theta)\, d\theta \\ &\geqslant B(p+1, d-1) \sin^{p+d-1} \theta_0 = \beta_{d,p}(\theta_0), \qquad 0 \leqslant \theta_2 \leqslant \pi/2 \end{aligned}$$

into the estimate of Theorem 2. This can be derived by induction on d: for $d = 2$, the assertion is

$$\int_0^{\theta_0} \sin^p \theta\, d\theta \geqslant \frac{\sin^{p+1} \theta_0}{p+1},$$

which is easily seen by comparison at 0 and ordering of derivatives. For the induction step, we use the same device: $\alpha_{d-1,p}(0) = 0 = \beta_{d+1,p}(0)$. A simple recursion of integrals yields

$$\alpha'_{d-1,p}(\theta_0) = (d-1)\, \alpha_{d-2,p}(\theta_0) \geqslant \beta'_{d+1,p}(\theta_0).$$

It remains to note that $\sin \theta_0 \geqslant \delta_\infty(K, L)/D$.

In the next corollary, we depart from viewing K and L symmetrically and obtain a local estimate.

COROLLARY 2. *Let $K, L \in \mathscr{K}^d$.* Then

$$C_p'' \cdot \left\{ \frac{\delta_\infty^{p+d-1}(K, L)}{[D_L + 2\delta_\infty(K, L)]^{d-1}} \right\}^{1/p} \leqslant \delta_p(K, L)$$

where $D_L = \mathrm{diam}(L)$, *and*

$$C_p'' = \left[\frac{B(p+1, d-1)}{B(\frac{1}{2}, (d-1)/2)} \right]^{1/p}.$$

Proof. Recall that for any two sets K and L, $\mathrm{diam}(K \cup L) \leqslant \mathrm{diam}(L) + 2\delta_\infty(K, L)$. Inserting this into the previous inequality yields the result.

We are now prepared to state that the spaces generated by the various metrics are closely related. It is natural to include a characterization of compact sets which essentially generalizes Blaschke's selection theorem to $p < \infty$.

THEOREM 3. *All of the δ_p metrics, $1 \leqslant p \leqslant \infty$, induce the same topology on $\mathscr{K}^d$ and yield complete metric spaces in which closed, bounded sets are compact.*

Proof. Theorem 1 and Corollary 2 imply that for fixed L and a sequence K_n, $\delta_\infty(K_n, L) \to 0$ iff $\delta_p(K_n, L) \to 0$. Accordingly, the generated topologies are the same.

The compactness and completeness statements are standard for δ_∞. For $p < \infty$ and compactness, observe that δ_p and δ_∞ yield the same compact sets and the same closed, bounded sets. For $p < \infty$ and completeness, it suffices to note that the closure of the set of points in a Cauchy sequence is bounded and hence compact.

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Note added in proof. Related results appear in [11].

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